Hiring, Firing and Infighting: A Tale of Two Companies*

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Abstract

We extend the hiring and firing framework of Shepp and Shiryaev (1996) to include infighting, and solve the profit-maximization problem using our numerical technique. With infighting, we find a smaller optimal firm size, and lowered firm value that stems from reduced operations and rapid profit-taking. Upon calibrating our model to match the cases of Levi Strauss & Co. and Microsoft, we find with Levi's, that our model matches media estimates of its loss in value fairly closely. With Microsoft, our model might explain its loss in value and poor decision-making, such as its recent acquisitions. The implications of our model for corporate governance might also help in reducing the current unemployment rate.

Keywords: Stochastic control, numerical methods, Brownian motion, corporate infighting

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1 Motivation and Introduction

In Shepp and Shiryaev (1996), the authors provide a simple stochastic model for the optimal size of staff of a firm. Although they find several interesting insights, they fail to take into account the effect of infighting on a company. Infighting between groups or individuals can have drastic effects on both, a firms hiring and firing policies, as well as its value. There are two examples in recent US corporate history that illustrate this fact:

1. In 1997, Levi Strauss & Co. was forced to lay off about a third of its workforce, when a plan to implement a team-based approach failed. *The Wall Street Journal* (May 20, 1998) states that this approach of the company "led to a quagmire in which skilled workers . . . found themselves pitted against slower colleagues, damaging morale and triggering corrosive infighting." As the company is privately owned, we do not have access to detailed data on its firm value, but *Fortune* (April 12, 1999) estimated that its value had shrunk from \$14 billion to \$8 billion in the period from 1996 to 1999, a loss of 42.85%. We do have data on revenues[1], and these have seen a steady decline from their 1997 peak of \$6.9 billion, to \$4.11 billion in 2009.

2. Former Microsoft executive Dick Brass wrote, in an op-ed piece in the *New York Times* (February 4, 2010), "At Microsoft, [internal competition] has created a dysfunctional corporate culture in which the big established groups are allowed to prey upon emerging teams, belittle their efforts, compete unfairly against them for resources, and over time hector them out of existence." The market capitalization for Microsoft peaked at the end of the year in 1999 at approximately \$500 billion, and was last recorded (as of this writing) at \$216 billion, about thirty-five percent of its 1999 peak. Arguably, it was this infighting that led to poor decision-making, leading to a drop in stock price. For instance, Microsoft acquired Skype – a company that was losing money at the time – at a price far higher than its actual worth[2]. In fact, Microsoft's infighting has become notoriously famous, even filtering into the popular culture with cartoons like the one shown in Figure 1.

While the intuition is clear on the relationship between corporate infighting, firm size and firm value; the theory is not. There is surprisingly little in the finance and economics literature that discusses this effect, much less trying to quantify it. In this paper, we try to fill this gap. We measure firm size by the number of employees, and firm value as the expected, discounted value of future profits. We extend the hiring and firing framework of Shepp and Shiryaev (1996) to include infighting and effectively answer the question, *what are the effects of infighting on hiring, firing and firm value*?

Not surprisingly, we find that infighting leads to reduction in overall firm value and size. When calibrated for Levi Strauss & Co., our model predicts that firm value should fall by 49.95%, which is fairly close[3] to the *Fortune* estimate (cited earlier) of 42.85%. In general, we

[1] All data in this paper on employee numbers and revenues are from the *Wolfram Alpha* database.

[2] See "Microsoft's Pricey Call on Skype," *The Wall Street Journal* (May 11, 2011).

[3] With other parameters, we can get the difference in value to be less than 1% away from the *Fortune* estimate. However, these parameters are realistic for the short-term but not the long-term. Please see the Results section (Section 3) for more details.

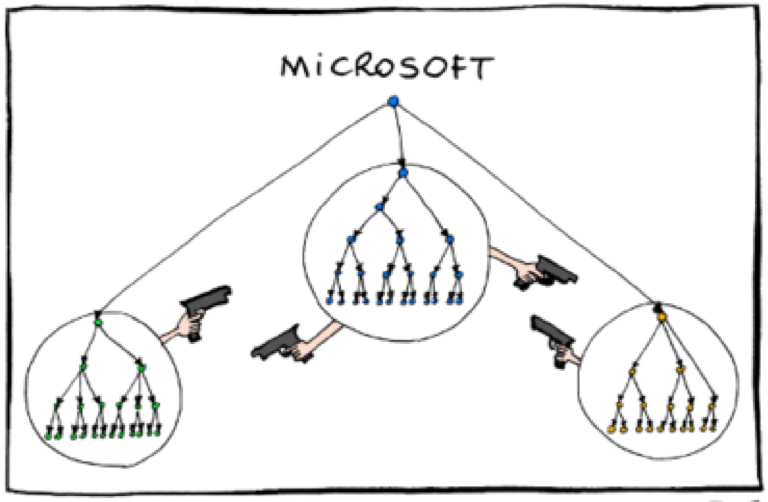

Figure 1: A cartoon by Manu Cornet (`www.bonkersworld.net`) depicting the current organizational structure at Microsoft.

find that in firms with infighting, it is optimal to take profits for lower levels of cash. There is, in fact, an inverse relationship between the level of infighting and the profit-taking threshold, and instead of reinvesting into the firm, managers quickly disburse any profits to owners. In a similar vein, operations for a firm reduce considerably, which might explain why firm value is also reduced. In fact, for a large part of their existence, firms with positive levels of infighting would find it optimal only to fire – *even when cash reserves are relatively high*.

When our model is calibrated to match Microsoft's parameters, it might explain why Microsoft has seen such a drastic reduction in its market capitalization: given its level of infighting it should, according to our model, be reducing firm size. It has, in fact, been steadily *increasing* its employee base and currently has about three times the number of employees it had in 1999 (90,000 versus 31,396). Because it has a sub-optimally high level of employees in the firm, given its high level of infighting, it is moving towards a lower optimal firm value, as our model would predict. If corporate governance focused on this issue of infighting, and created a less hostile work environment, this would result in firms optimally hiring, thus increasing value for themselves, and reducing the national unemployment rate.

To model infighting, we turn to nature, and more specifically, mathematical models of population dynamics, such as the Verhulst-Pearl diffusion[4]. These models account for competition and infighting in animal herds, and show that this has a controlling effect on their size. Essentially, two animals in the herd, frequently dominant males, will fight with each other over a limited resource; the loser will die, thus keeping the size of the herd in check. Guo (2002), adapts the population density model of Alvarez and Shepp (1998) to model infighting in a corporate context, and extends Shepp and Shiryaev (1996), though her model does not include hiring and firing costs (k_+ and k_-). Her model includes a term, α, that represents the effect of competition

[4]See May (1973).

or infighting. We extend, Shepp and Shiryaev (1996), and Guo (2002), to create a model for cash reserves that includes hiring, firing and infighting as follows:

$$dX_t = U_t\left(\mu dt + \sigma dW_t\right) - k_- d_- U_t - k_+ d_+ U_t - \alpha U_t^2 dt - dZ_t \tag{1}$$

Here, X_t are the company's cash reserves, U_t is the number of employees, Z_t are the profits taken out[5], W_t is a standard Brownian or Wiener process, and k_- and k_+ are the one-time costs of hiring (e.g., relocation) and firing (e.g., severance pay), respectively, and the (μ, σ) pairs represent risk-return characteristics of the current investment. This investment could, with equal likelihood, be a project of some sort (e.g., manufacturing, expansion, etc.) or a portfolio of financial securities. Finally, α is a competition or infighting term, $0 < \alpha \leq 1$. When $\alpha = 0$ (or no infighting) the model reduces to the one in Shepp-Shiryaev.

Our objective is to maximize the expected discounted dividends (or profits for a privately-owned firm), dZ_t over the life of the firm:

$$V(x) = \sup_{Z_t, U_t} \mathrm{E}_{x,u} \int_0^\infty e^{-rt} dZ_t, \text{where } x = X_0 \text{ and } u = U_0 \tag{2}$$

Mathematical tractability makes this problem difficult to solve in closed form, and to overcome this, we use a numerical technique to solve this problem. This was first seen in Sheth et al. (2011), where the authors show the optimality of risk taking under financial distress. In testing modifications of the technique to solve the Shepp-Shiryaev problem, which has a known solution, we get errors that are in the range of 10^{-4} (please see the Appendix for details), and we are confident of the results of using this technique to solve the problem under consideration here. Furthermore, this is a "1.5-state" problem, since while X_t is stochastic, U_t is not. As a result, we approach this by solving for a stochastic X_t, $\forall\, U_t$. Thus, we have separate employment and dividend policies for each level of employment.

2 Methodology

We start by creating a revenue process, Y_t, such that:

$$Y_t = \overline{V}(X_t, U_t)e^{-rt} + \int_0^t e^{-rs} dZ_s \tag{3}$$

This process represents the future and past discounted dividends, and $\overline{V}(X, U)$ is a guess to the true value function such that $EdY_t \leq 0$, or the process is expectation-decreasing (a supermartingale). In order to proceed we use the following Lemma from Shepp and Shiryaev (1996):

Lemma 1. *Suppose $\overline{V}(x, u) \geq 0$ is such that for any choice of U_t or Z_t, the process Y_t is expectation-decreasing. Then $V \leq \overline{V}$.*

Proof. Note that $EY_0 \leq EY_\infty$, but $EY_0 = \overline{V}(x, u)$ and $EY_\infty = E\int_0^\infty e^{-rs} dZ_s$. Since this holds for every U_t and Z_t, it shows that $V(x, u) \leq \overline{V}(x, u)$. □

[5]For a publicly traded company, these could be interpreted as dividends.

Lemma 1 shows us that $V \leq \overline{V}$. Since our guess is an upper-bound on the optimal solution to the problem of maximizing dividends, the true solution will be the smallest upper-bound. Thus, if we minimize $\overline{V}$, we will get the true solution, and to do this, we use linear programming.

Before we do this however, we must find the conditions under which the process Y_t is a supermartingale. For that, we need that $EdY_t \leq 0$. To that end, we first find dY.

$$\begin{aligned} dY_t &= d\overline{V}e^{-rt} + d\left(\int_0^t e^{-rs} dZ_s\right) \\ &= d\overline{V}e^{-rt} - re^{-rt}\overline{V}dt + e^{-rt}dZ_t \end{aligned} \tag{4}$$

We apply $d\overline{V} = \frac{\partial \overline{V}}{\partial X_t} dX_t + \frac{1}{2}\frac{\partial^2 \overline{V}}{\partial X^2}(dX_t)^2 + \frac{\partial \overline{V}}{\partial U_t} dU_t$ (Itô's Lemma) to get $d\overline{V}$.

And since:

$$\begin{aligned} dX_t &= U_t(\mu dt + \sigma dW_t) - k_- d_- U_t - k_+ d_+ U_t - \alpha U_t^2 dt - dZ_t \\ \left(dX_t^2\right)^2 &= \sigma^2 U_t^2 dt \\ E\left(\overline{V}_X U_t \sigma dW_t\right) &= 0 \end{aligned}$$

We get that:

$$\begin{aligned} EdY_t = e^{-rt}\{&(1 - \overline{V}_X)dZ_t + (\overline{V}_U - k_+\overline{V}_X)d_+U_t + (-\overline{V}_U - k_-\overline{V}_X)d_-U_t \\ &+ (-r\overline{V} + (\mu U_t - \alpha U_t^2)\overline{V}_X + \frac{\sigma^2}{2}U_t^2\overline{V}_{XX})dt\} \end{aligned} \tag{5}$$

where $\overline{V} = \overline{V}(X_t, U_t)$; $\overline{V}_X = \frac{\partial \overline{V}}{\partial X}$; $\overline{V}_{XX} = \frac{\partial^2 \overline{V}}{\partial X^2}$; and $\overline{V}_U = \frac{\partial \overline{V}}{\partial U}$.

So to have $E_{x,u}dY_t \leq 0$, we need that:

1. $M\overline{V}(x,u) = 1 - \overline{V}_x \leq 0$
2. $H\overline{V}(x,u) = \overline{V}_u - k_+\overline{V}_x \leq 0$
3. $F\overline{V}(x,u) = -\overline{V}_u - k_-\overline{V}_x \leq 0$
4. $L\overline{V}(x,u) = -r\overline{V} + (\mu u - \alpha u^2)\overline{V}_x + \frac{\sigma^2}{2}u^2\overline{V}_{xx} \leq 0$

Once the above constraints are fulfilled, we get a $\overline{V}$ such that $V \leq \overline{V}$ (by Lemma 1). And by minimizing $\overline{V}$, we get the optimal solution. To do this, we use linear programming to solve the following minimization problem:

$$\begin{aligned}
&\min \overline{V} \\
&\text{subject to} \\
&M\overline{V}(x,u) : 1 - \overline{V}_x \leq 0 \\
&H\overline{V}(x,u) : \overline{V}_u - k_+\overline{V}_x \leq 0 \\
&F\overline{V}(x,u) : -\overline{V}_u - k_-\overline{V}_x \leq 0 \\
&L\overline{V}(x,u) : -r\overline{V} + (\mu u - \alpha u^2)\overline{V}_x + \frac{\sigma^2}{2}u^2\overline{V}_{xx} \leq 0
\end{aligned} \tag{6}$$

We refer to the constraints in (6) as the behavioral constraints. When the M, H, F and L constraints are strictly equal to zero, the manager, respectively, takes profits, hires, fires and operates the firm using the corresponding (μ, σ) pair.

The discretization process, and the process of constraint matrix creation, is outlined in the Appendix.

3 Results

We find that there is a loss in both, maximum potential firm value, as well as optimal number of employees in the firm, when the infighting parameter, $\alpha \neq 0$. First however, we discuss our logic in choosing the model parameters.

3.1 Calibrating Model Parameters

There are six parameters whose value must be decided: μ, σ, r, k_+, k_- and α. For the one-time hiring and firing costs, k_+ and k_-, we are restricted by their upper bounds as specified in Shepp and Shiryaev (1996): $\xi(\mu, \sigma)$ and $\eta(r, \mu, \sigma, k_+)$, respectively, for the hiring and firing costs (see the Appendix for details). For the risk-free rate, r, we look at one-year US Treasury bills, and we find that these have a historical annual geometric mean of about 5%, which we use. For the (μ, σ) pairs, we look at the annualized geometric means and standard deviations of monthly stock returns of the respective company stock, since it went public. For Microsoft, which is publicly traded, this data is easily available, and works out to be $\mu = 25\%$, and $\sigma = 36\%$, from 1986 to the present. Since Levi Strauss & Co. is private, we look at four comparable publicly-traded apparel manufacturing firms instead[6], and take the arithmetic average of their geometric means and standard deviations over the lives of the four stocks. For these companies, $\mu = 20\%$, and $\sigma = 45\%$[7]. We calibrate α based on the effect infighting has had on both companies:

[6] We used Under Armor (UA), Polo Ralph Lauren (PL), Oxford (OXM) and Gildan (GIL) that went public in 2005, 1997, 1987 and 1998, respectively.

[7] If we use $\mu = 75\%$ and $\sigma = 100\%$, we get a loss in value of 41.95%, which is significantly closer to the *Fortune* estimate of 42.85%. It should be noted that there were some apparel companies that saw such returns in recent years, e.g., Under Armor had a return of 78.58% in the last year, and Oxford had a return of 74.67%, but these are an anomaly, and it is unlikely that they will persist over the long term.

1. For Levi Strauss & Co., we assume a one-third loss of employees compared to the no-infighting case. This was the case from 1997 to 1998, when Levi's lost exactly 33.33% of its employees (the number went from 30,000 to 20,000).

2. For Microsoft, we do not have estimates of firm value, so we work under the assumption that the loss in market capitalization down to 35% of its peak is a proxy for its loss in firm value.

Once we have the α that matches these two criteria for their respective (μ, σ) pairs, we then set k_+ and k_- to be below their upper-bounds. These costs are averages, and we interpret them as means for a distribution. For instance, some employees will receive a higher severance package than others. We start by looking at the base case from Shepp and Shiryaev (1996), i.e., $\alpha = 0$ (no infighting), to illustrate the models basic workings, and to test our numerical technique.

3.2 The Base Case

For the base case, we use the (μ, σ) pair for Levi Strauss & Co., though we had similar results for several other parameters, including those of Microsoft. We find that there exists is a maximum employment level for the firm, M, and the value function either stays the same or decreases[8] for all values of employment greater than M. We interpret this as being the optimal level of employees for the firm, as there is no other level of employment for which the value function is higher – this is the maximum-value employment level. For the set of parameters used in the base case, $M = 4.675$, where in our state-space, $u_{\min} = 0$ and $u_{\max} = 5$. This is the point after which the value function stops increasing in Figure 2.

We then create an entire schematic for each level of x (cash) and u (employment). Based on which constraint, i.e., hiring, firing, operating or profit-taking, is equivalently zero for that corresponding x and u, it tells us which is the optimal action. In Figure 3, the magenta region is the firing region, so for any value of x and u that falls in this region, it is optimal for the firm to fire. Similarly, the blue region is the operational region, where the firm operates and reaps the rewards of the project offering it the risk-return profile of (μ, σ). Also, the green region is the hiring region, and the black region is when it should take profits. As in Shepp and Shiryaev (1996), $dt = 0$ in the hiring, firing and profit-taking regions, so all of these are done instantaneously; $dt > 0$ in the operating region only.

In the Shepp-Shiryaev framework, there exists a threshold, a, and as long as $X_t > a$, it is optimal to instantaneously take as profits, all cash that is greater than this threshold. We do have an explicit solution for this threshold (from Radner and Shepp (1996)):

$$a = \frac{\log(\alpha_1/\alpha_2)^2}{(\alpha_2 - \alpha_1)} \tag{7}$$

where α_1 and α_2 are the negative and positive roots, respectively, of the quadratic: $-r + \mu x + \frac{1}{2}\sigma^2 x^2 = 0$. We find that a varies as u increases, and there are points where it is *never* optimal to take profits, instead putting all the money into the firm (for hiring, for instance). Furthermore,

[8] We calculate this level by taking the average of $V(x), \forall u$.

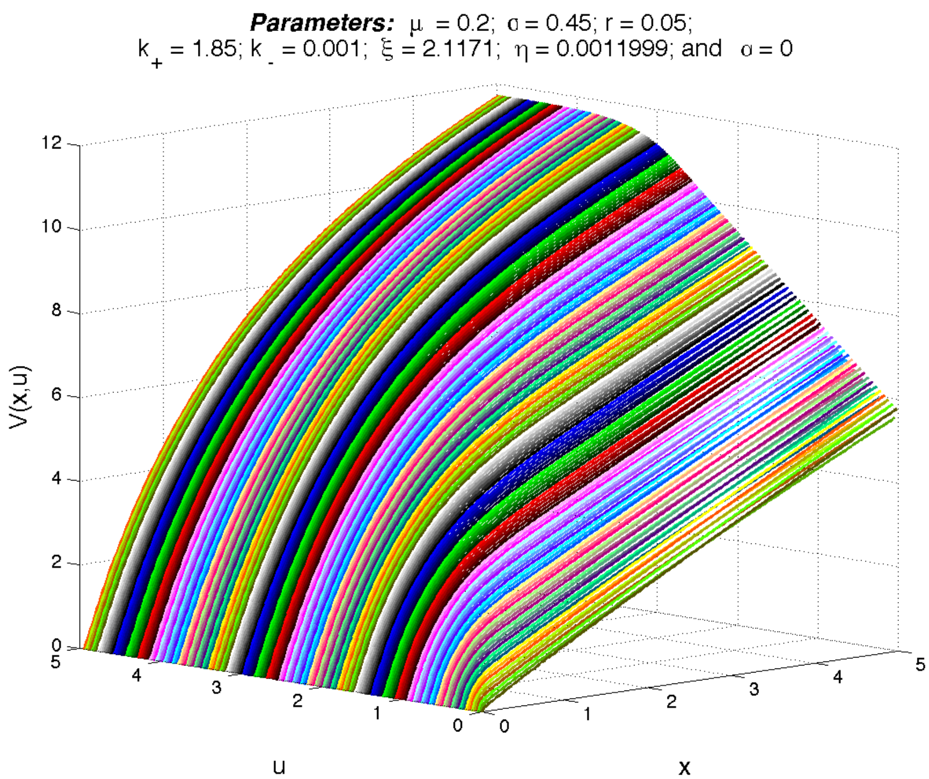

Figure 2: For this set of parameters, $V(x,u)$ increases up to $u = 4.675$, and stops increasing after that. This is the firms value-maximizing optimal employment level.

we find that a also varies with α, and it becomes optimal to take profits sooner when there is infighting within the firm. For $\alpha = 0$ in Figure 3, we see that when $u \approx 2.5$, there is no profit-taking threshold, i.e., it is never optimal to take profits for all values of x in this region. When we move into the operating region however, it becomes optimal once again to give out profits at $x = x_{\max} = 5$.

3.3 Levi Strauss & Co.

As stated above, we use four comparable apparel companies for Levi's, to get $\mu = 20\%$ and $\sigma = 45\%$. Using these values, we first get the optimal employment rate with $\alpha = 0$. At this point, we assume a 33.33% loss in employees, and we calibrate α so that it results in a loss of optimal employment level down from $u = 4.675$, to $u = \frac{4.675}{3} = 1.55$. The results are in Figure 4.

We find that the company spends most of its time firing, and this matches up with the reality – Levi's went from having 37,700 employees in 1996, to 9,635 in 2005, when it started hiring again (see Figure 5). In addition, we also see that the profit-taking region increases drastically, as compared with the case when there is no infighting. While we do not have the data to say whether there was increased profit-taking in reality, we can say that owing to its early profit-taking and lowered operations as seen in our model, the company saw a consistent loss in revenues across this period, as is seen in Figure 5. Finally, since the company has been behaving

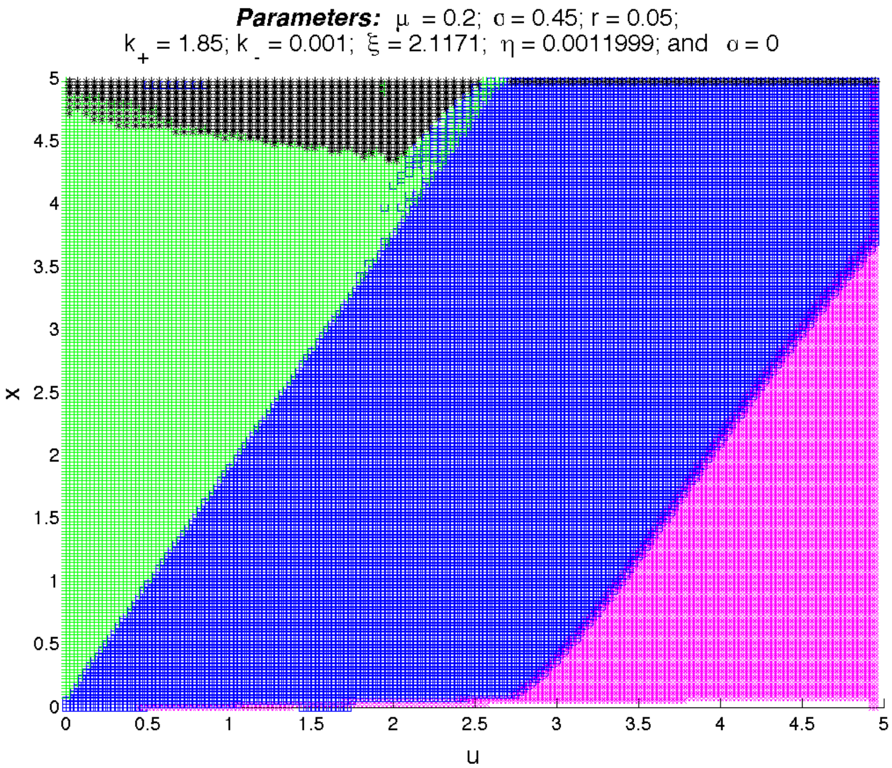

Figure 3: The entire schematic $\forall x, u$ when there is no infighting, or $\alpha = 0$. If a firm is in the magenta region, it is optimal to fire; in the green region it is optimal to hire; in the black region the firm takes profits; and in the blue region it operates normally. Note that $dt = 0$ everywhere except in the operating region, so the firm keeps hiring, firing or taking profits instantaneously until it comes back into the operating region, where $dt > 0$.

optimally according to our model, it has seen an average increase in revenues of 1.98% per year, from 2005 to 2008[9]. The layoffs and lowered operations caused by the infighting decreased firm value for the company, but according to our model, this was optimal behavior which resulted in decreased infighting and increased profits in the long run.

3.4 Microsoft

For Microsoft, we use values based on its stock price of $\mu = 25\%$ and $\sigma = 36\%$. We also assume that its loss in market capitalization is a good proxy to its loss in value, and we calibrate the infighting parameter so that the company goes down to 35% of its no-infighting value. We calculate the value as an average for each level of employment, and for $\alpha = 0$, the value is 14.5306. We calibrate α so that the value decreases down to $14.5306 \times 35\% = 5.086$. This happens at $\alpha = 3.9713 \times 10^{-4}$. For this level of infighting, we find that the company should not hire at all, and instead spend most of its time firing, or operating (see Figure 6).

[9]Revenue growth from 2008-2009 (the last year for which we have data) was negative, but this can be attributed to the financial crisis.

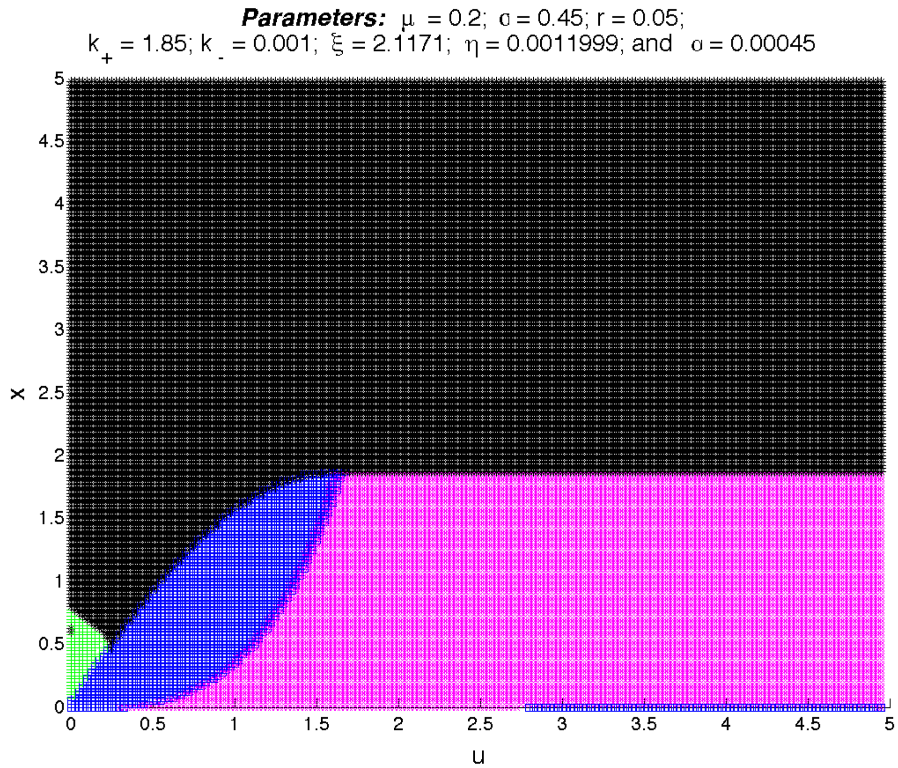

Figure 4: The entire schematic $\forall x, u$ for Levi Strauss & Co. The infighting parameter is calibrated so that the company experiences a loss of 33.33% of its employees with no infighting. This happens at $\alpha = 4.5 \times 10^{-4}$. Note that the company does very little hiring, and for a large part of its existence, it is mostly firing its employees (the magenta region), or taking profits (the black region).

Instead, the company has mainly spent the last fifteen years or so hiring more and more employees: the numbers have increased from 17,801 in 1995 to aproximately 90,000 in 2011. Also, revenues have been increasing at an average rate of 11.6% per year, but they are increasing at a *decreasing rate* – see Figure 7. This can be explained by our model, as instead of firing and reducing firm size to thereby increase firm value, Microsoft has been increasing firm size, thus reducing its optimal firm value, given its high level of infighting. By decreasing firm size, not only will Microsoft move towards increasing optimal firm value, but also will increase the optimal value as fewer employees also means a reduction in the infighting parameter value, as we saw with Levi Strauss & Co.

Increased profit-taking is synonymous with increased dividends in the Shepp-Shiryaev framework, and it is well known that Microsoft gave out a $3 per share special dividend in 2004 for the first time in the company's history, after which it started to give out dividends on a regular basis. These have increased from $0.08 per share in 2004, up to $0.20 per share in 2011. In this sense, the company is behaving optimally according to our model, and we will see if this has any effect on long run value into the future.

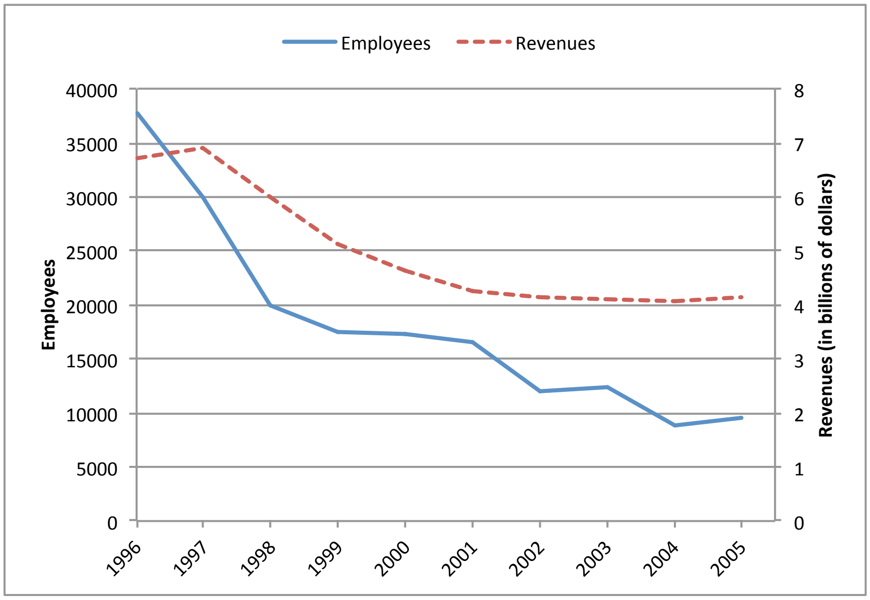

Figure 5: Number of employees and revenues for Levi Strauss & Co. from 1996 to 2005 (inclusive). Both show a steady decrease as a result of the infighting, as our model predicts.

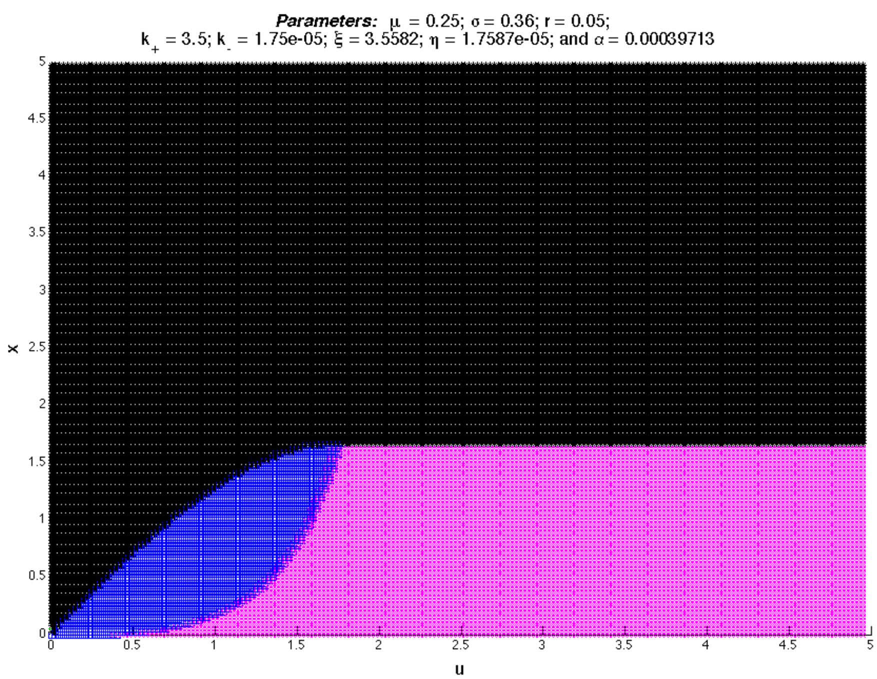

Figure 6: The entire schematic $\forall x, u$ for Microsoft. The infighting parameter is calibrated so that the company experiences a loss down to 35% of its no-infighting firm value. This happens at $\alpha = 3.9713 \times 10^{-4}$. Note that the company does **no** hiring, and for a large part of its existence, it is mostly firing its employees (the magenta region), or taking profits (the black region).

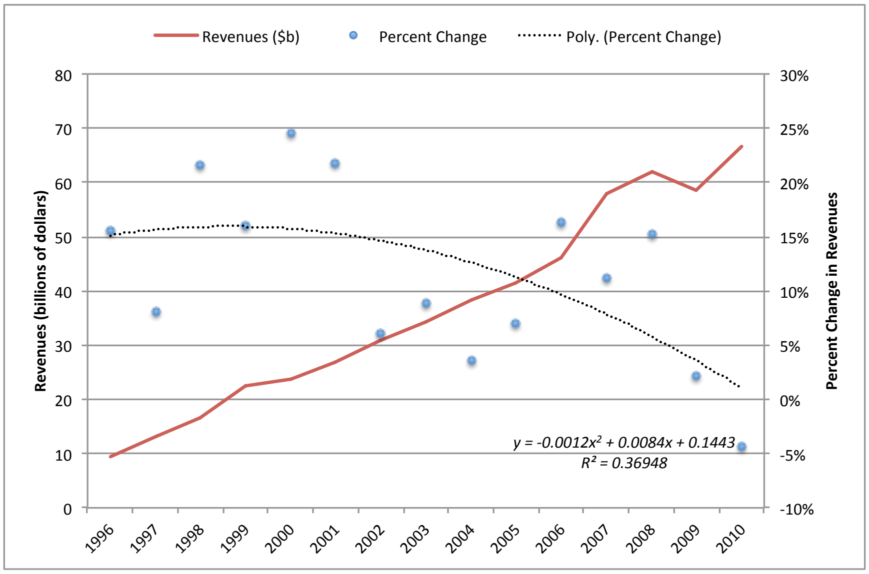

Figure 7: Absolute revenues and the year-to-year percent change in revenues for Microsoft from 1996 to 2010 (inclusive). While revenues have been increasing, the percent change in revenues has been on a steady decline, as depicted by the polynomial trend line also shown above. Microsoft has been hiring, instead of firing optimally as it should be doing, and so we expect to see a decrease in firm value, as our model predicts.

4 Conclusion and Implications

We find, not surprisingly, that infighting within a firm will affect both: the value of the firm, as measured by maximum future, expected, discounted profits; as well as the optimal size of the firm as measured by the number of employees. We find that infighting will result in short-sighted, profit-taking behavior which results in very little money being put back into the company. We also find that infighting results in drastically lowered operations.

We find that for Levi's our model predicted the company's hiring and firing policies over the decade or so following the large levels of infighting. While there were certainly other factors affecting the drop in revenues and layoffs, in the immediate years following the 1997 it is clear that the team-based approach that the company tried (and failed) to implement did play a role in the loss of company value and employees. Furthermore, as the company has been behaving as our model predicts, i.e., in an optimal value-maximizing fashion, it has seen an uptick in revenues in recent years.

For Microsoft, our model states that given their current level of infighting, they should be laying off employees, not hiring – which would result in a loss of firm value. Certainly, this is reflected in the market capitalization of the company, and though it is seeing an increase in revenues, these are increasing at a decreasing rate. The lowered operations, and lack of optimal hiring (given their infighting levels) might explain their poor decision-making when it came to recent acquisitions. Our model predicts increased dividends for increased infighting and, in this regard, Microsoft is behaving optimally. Perhaps this will have a positive effect on its value in the future.

Finally, it is surprising that given its potentially devastating effect on a firm, current policy does not take into account the nature of infighting when it comes to the hiring and firing policy of a firm. Perhaps if both, politicians and corporate boards, work towards curbing infighting within firms, this would reduce the overall value of α for all firms, pushing firms into a region where it becomes optimal for them to hire. This is one way in which the current unemployment rate can be improved.

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Appendix

Discretization

Since we are trying to optimize (with linear programming) a continuous-time problem, we must first discretize it. The variables x and u are discretized on suitable intervals, $x \in [0, x_{max}]$, $u \in [0, u_{max}]$, with $x = ih$, $u = jh$ where $i, j = 1, 2, \ldots, n_0$ and $h_x = x_{max}/n_0$ and $h_u = u_m ax/n_0$. The $O(h)$ and $O(h^2)$ approximations to the derivatives were used:

$$\begin{aligned}
\overline{V}(x_i, u_j) &= \overline{V}_{i,j} \\
\overline{V}_x(x_i, u_j) &= \frac{\overline{V}_{i+1,j} - \overline{V}_{i-1,j}}{2h_x} = \frac{\overline{V}_{i+2,j+2} - \overline{V}_{i,j+1}}{2h_x} \\
\overline{V}_{xx}(x_i, u_j) &= \frac{\overline{V}_{i+1,j} - 2\overline{V}_{i,j} + \overline{V}_{i-1,j}}{h_x^2} = \frac{\overline{V}_{i+2,j+1} - 2\overline{V}_{i+1,j+1} + \overline{V}_{i,j+1}}{h_x^2} \\
\overline{V}_u(x_i, u_j) &= \frac{\overline{V}_{i,j+1} - \overline{V}_{i,j-1}}{2h_u} = \frac{\overline{V}_{i+1,j+2} - \overline{V}_{i+1,j}}{2h_u}
\end{aligned} \tag{8}$$

By imposing the linear constraints at n_0 interior points, we get a total of $4n_0$ constraints and $(n_0 + 2)^2$ unknown variables $\overline{V}_{i,j}$, where $i, j = 1, 2, \ldots, n_0$. Also, h_x is the size of the smallest interval in the grid for x and h_u is its counterpart in the grid for u. For convenience, we keep $h_x = h_u$.The finite-dimensional problem is stated as:

$$\begin{aligned}
&\min c^T v \\
&\text{s. t.} \\
&Av \leq b \\
&v_{1,j} = 0, \forall j
\end{aligned}$$

where v is an unknown vector of length $(n_0+2)^2$, A is a $4n_0^2 \times (n_0+2)^2$ matrix, c is $1 \times (n_0+2)^2$ and b_j is a vector of length n_0^2. We kept $c_j = 1, \forall j$[10].

The Constraint Matrix, A

For clarification, we provide some basics on the constraint matrix, since the value function has two variables - the cash reserves, x, and the employment level, u. In the notation used above, i represents the cash reserves, and j represents the employment level.

With one variable, we only have v_i, where i is the column in the constraint matrix, A. Each iteration, therefore, has a row. With two variables, we have $v_{i,j}$. Each iteration still, however, has one row. In this case, both i and j are columns, but for each i, we have a corresponding j. Since the result of the linear program is a vector, we can linearly produce a j for every i. In

[10]We get similar results for $c_i = 1$ for arbitrary i and $c_j = 0, \forall j \neq i$. This is a consequence of our numerical technique. By minimizing one arbitrary element of v, we get the solution to our problem.

other words, the first $(n_0 + 2)$ set of rows in the linear programming results would be for $j = 1$, the second $(n_0 + 2)$ set of rows would be for $j = 2$, and so on. To simplify things, we let $i = j$ and so we end up with $(n_0 + 2)^2$ rows.

Since we have four constraints, (see equations (6)), the constraint matrix will have $4n_0^2$ rows - for each i there are four constraints, each with n_0 rows, and the same for each j. Consequently, the matrix will have $(n_0 + 2)^2$ columns, which would also explain the number of rows in the results vector.

Verification of the Shepp and Shiryaev (1996) Results Using Our Numerical Method

We verify the hiring and firing linear programming results by comparing them with the Radner and Shepp (1996) analytical solution for when $k_+ = k_- = 0$. However (as mentioned earlier), the results do not match up when this is the case. In fact, Shepp and Shiryaev, in their model, explain that there are upper bounds on the hiring and firing costs, viz. ξ and η, respectively. In Shepp and Shiryaev (1996), the manager only hires when $k_+ < \xi$, and $k_- < \eta$. If we violate these boundaries, there is no hiring and firing in their model. However, in our model, we find that we can verify the linear programming results fairly well *only* if we violate the hiring and firing cost boundaries.

The analytical solution provided in Radner and Shepp (1996) for the $n = 1$ case, is:

$$V(x) = \begin{cases} A\left(e^{\gamma_+ x} - e^{\gamma_- x}\right) & \text{for } 0 \le x \le a \\ V(a) + x - a & \text{for } a < x \end{cases} \tag{9}$$

where,

$$A = \frac{1}{\gamma_+ e^{\gamma_+ a_0} - \gamma_- e^{\gamma_- a_0}}$$
$$a = \frac{1}{\gamma_+ - \gamma_-} \ln\left(\frac{\gamma_-}{\gamma_+}\right)^2$$
$$\gamma\pm = -\mu \pm \frac{\sqrt{\mu^2 + 2r\sigma^2}}{\sigma^2}$$

γ_+ and γ_- are the positive and negative roots of the quadratic $\frac{1}{2}\sigma^2 x^2 + \mu x - r = 0$; and a is the dividends threshold.

This is the solution, when the cash reserves are given by the Radner and Shepp (1996) stochastic process:

$$dX_t = \mu dt + \sigma dW_t - dZ_t \tag{10}$$

with W being a standard Brownian motion.

The Shepp and Shiryaev (1996) cash reserves are given by:

$$dX_t = U_t\left(\mu dt + \sigma dW_t\right) - k_- d_- U_t - k_+ d_+ U_t - dZ_t \tag{11}$$

If we are at time, $t = 0$, then $X_0 = x$, and $U_0 = u$. Further, if we assume the hiring and firing costs, $k_+ = k_- = 0$, then (11) becomes:

$$dx = (\mu u)dt + (\sigma u)dW_t - dZ_t \tag{12}$$

This is, essentially, the Radner and Shepp (1996) model with one ($n = 1$) policy set, $(\mu u, \sigma u)$. And the analytical solution to this should match up with the Shepp and Shiryaev (1996) case for the corresponding level of u. However, as it turns out, this is not the case. Figure 8 below, displays the results for the analytical solution to the Radner-Shepp model with $\mu = 0.5$, and $\sigma = 0.65$, compared with the Shepp-Shiryaev model with the same parameters, and $u = 1$. The mean error is large, and has absolute value ≈ 32.

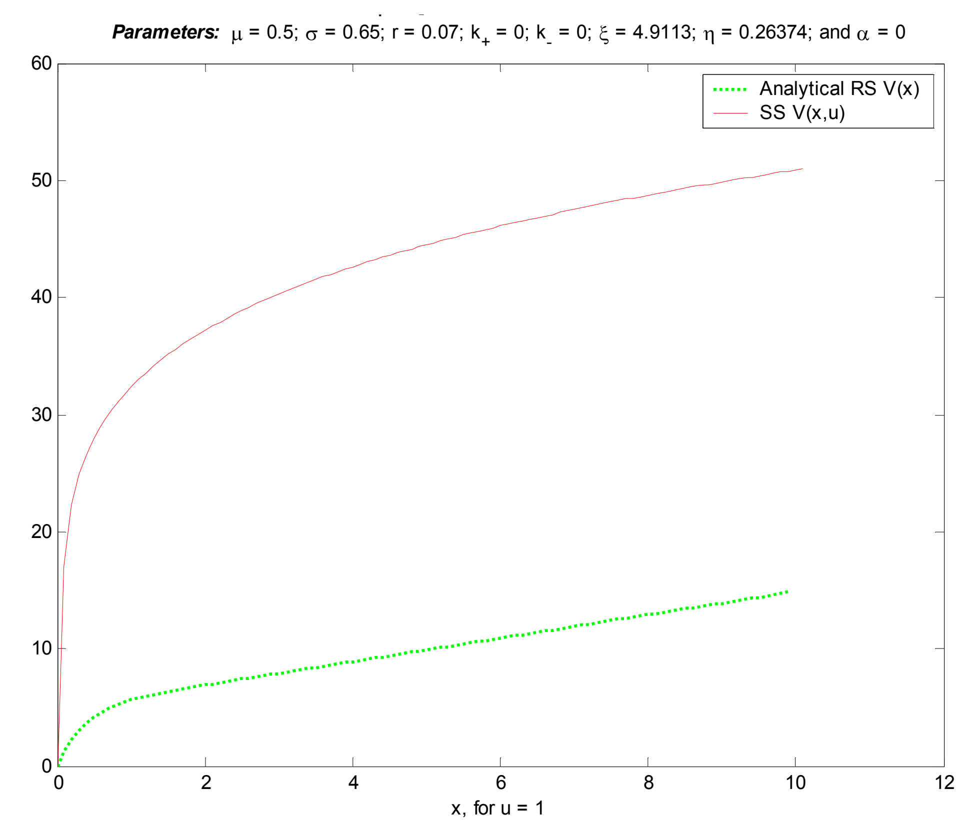

Figure 8: A comparison between the analytical solution to Radner-Shepp for the case with $\mu = .5$ and $\sigma = .65$, and the Shepp-Shiryaev case for the same parameters, with $u = 1$ and $k_+ = k_- = 0$. The mean error is clearly large and has absolute value ≈ 32.

For the (μ, σ) pair above, we have that, $\xi = 4.9113$, and $\eta = 0.2637$. As stated in Shepp and Shiryaev (1996), we only need to set $k_+ > \xi$ and $k_- > \eta$, to prevent any hiring and firing from taking place. As per the theory, this would make it prohibitively expensive to do so. We set $k_+ = 35$, and $k_- = 7$, to make these costs extremely large, and upon running the linear program once again, we find that the mean error is reduced to 10^{-4}. Figure 9 below, shows the results of increasing the hiring and firing costs, compared with the original Radner-Shepp results from Figure 8.

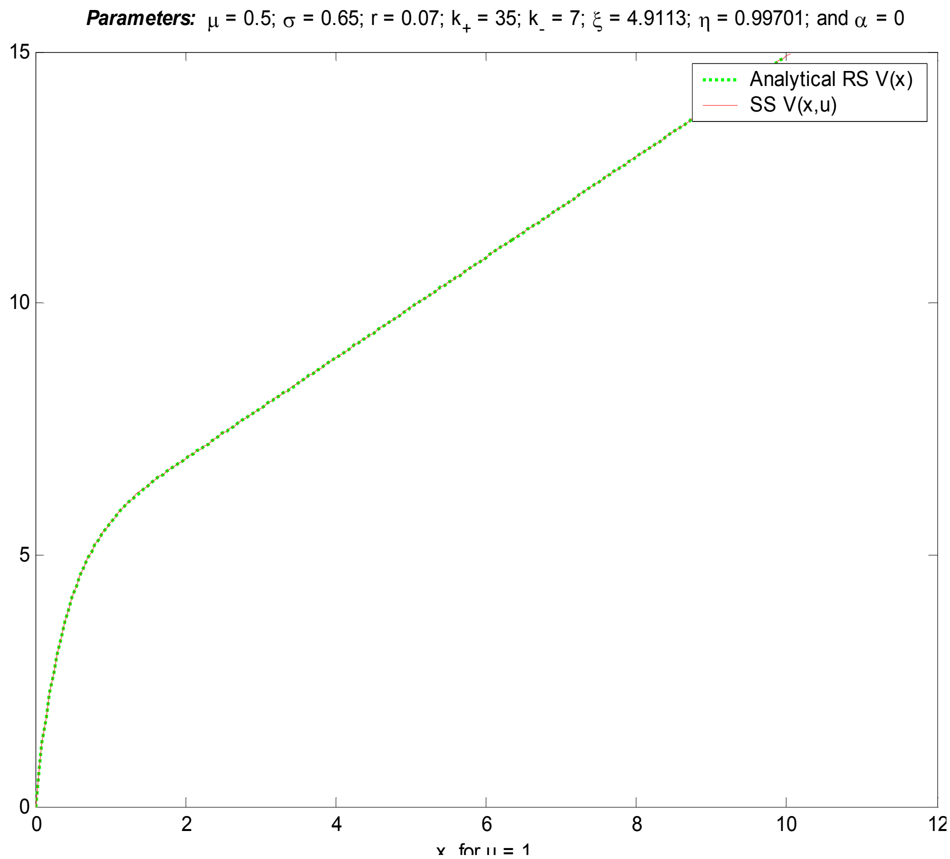

Figure 9: A comparison between the analytical solution to Radner-Shepp for the case with $\mu = .5$ and $\sigma = .65$, and the Shepp-Shiryaev case for the same parameters, with $u = 1$. The hiring and firing costs are greater than their acceptable boundaries, $k_+ = 35 > \xi = 4.9113$ and $k_- = 7 > \eta = 0.2637$. The mean error goes down from having absolute value ≈ 32, down to 10^{-4}.

The question then arises: *Why does the analytical solution of Radner-Shepp match up with Shepp-Shiryaev, when the hiring and firing costs are non-zero and exceed their upper bounds?* The reasoning is simple. It is because when the hiring and firing costs are zero, the manager has carte blanche to hire and fire as she pleases. There are no real restrictions placed on her, with regards to optimal employment. However, if we increase the hiring and firing costs until they become prohibitive, we find that the manager will never hire or fire just like in the case of Radner-Shepp, which has no hiring or firing. In this case we see that the Shepp-Shiryaev linear programming output matches up fairly well with the analytical solution of Radner-Shepp.

Figure 10, below shows the comprehensive set of hiring and firing constraints for all values of employment or u. Since the hiring and firing costs exceed their upper bounds, viz. $k_+ > \xi$ and $k_- > \eta$, it is never optimal to hire or fire. Thus, the constraints are always non-binding and negative.

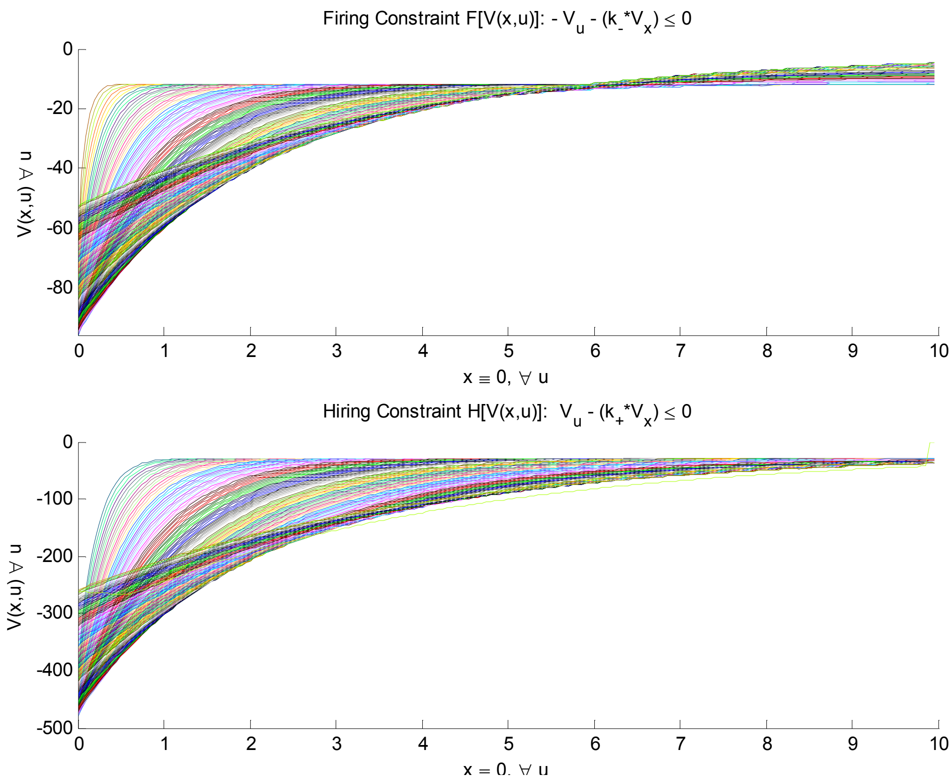

Figure 10: The comprehensive set of hiring and firing constraints for all values of employment or u. Since the hiring and firing costs exceed their upper bounds, viz. $k_+ > \xi$ and $k_- > \eta$, it is never optimal to hire or fire. Thus, the constraints are always non-binding and negative.

Bounds on Hiring and Firing Costs

For the one-time hiring and firing costs, k_+ and k_-, we are restricted by their upper bounds as specified in Shepp and Shiryaev (1996): $\xi(\mu, \sigma)$ and $\eta(r, \mu, \sigma, k_+)$, respectively, for the hiring and firing costs. These are given explicitly in Shepp and Shiryaev (1996):

$$\xi(\mu, \sigma) = \frac{\mu}{r} - a \tag{13}$$

$$\eta(r, \mu, \sigma, k_+) = a - \bar{b} - \bar{\theta}\frac{R_0(a - \bar{b})}{R_1(a - \bar{b})} \tag{14}$$

where,

$$R_i(x) = \alpha_2^i e^{\alpha_2 x} - \alpha_1^i e^{\alpha_1 x}$$
$$\bar{\theta} = \frac{R_1(\bar{b})}{R_0(\bar{b})}(\bar{b} + k_+)$$

Also, $\bar{b}$ can be calculated from $g(x) = \frac{R_1(x)R_0(x)}{\left(R_1^2(x) - R_0(x)R_2(x)\right) - x}$, for $0 \le x \le a$, since $g(\bar{b}) = k_+$ (see Figure 11).

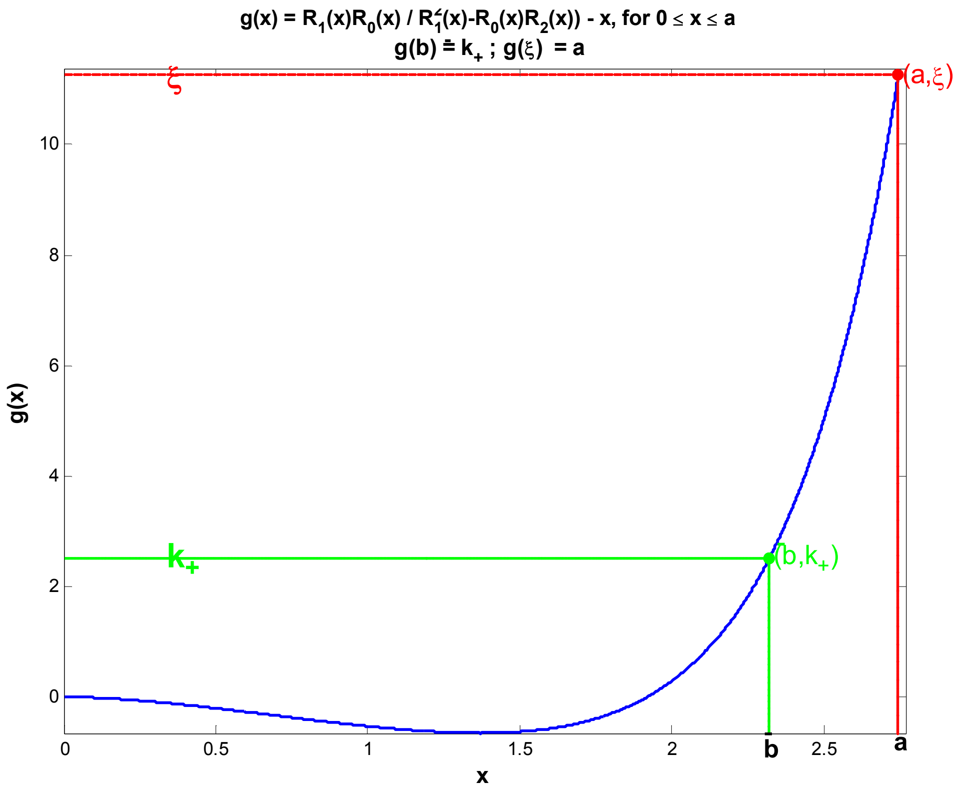

Figure 11: The graph of $g(x) = \frac{R_1(x)R_0(x)}{\left(R_1^2(x) - R_0(x)R_2(x)\right) - x}$. for $0 \le x \le a$, similar to the graph in Shepp and Shiryaev (1996), pp. 1531. We can ascertain the value of $\bar{b}$ from this graph since $g(\bar{b}) = k_+$.

We also have that a is the dividends (or profit-taking) threshold from Radner and Shepp (1996), for one policy ($n = 1$), and is given by:

$$a = \frac{\log(\alpha_1/\alpha_2)^2}{(\alpha_2 - \alpha_1)} \tag{7}$$

where α_1 and α_2 are the negative and positive roots, respectively, of the quadratic: $-r + \mu x + \frac{1}{2}\sigma^2 x^2 = 0$.